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# Subharmonic Solutions for Second Order Differential Systems

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Ink this paper, we consider the existence of subharmonic solutions for the problem $u_{tt} + G'(u) = f(t)$, where $G: R^N \to R$ is not necessarily convex and $f: R \to R^N$ is periodic with minimal period $T > 0$. © 1995 Academic Press, Inc.

## 1. INTRODUCTION

In this paper, we consider the system of second order differential equations of the form

$$\text{(P)} \qquad u_{tt} + G'(u) = f(t),$$

where $G \in C^2(R^N; R)$ and $f: R \to R^N$ is a continuous function with minimal period $T > 0$. Here $G'$ denote the gradient of $G$.

Our purpose in this paper is to show the existence of subharmonic solutions of (P). That is, we seek for a sequence $\{u_{k_i}\}$ of solutions of (P) where $u_{k_i}$ is a periodic function with minimal period $k_i T$ and $k_i \to \infty$, as $i \to \infty$.

The existence of subharmonic solutions of (P) has been studied by many authors [1–6, 9, 11, 13] (see [5] for further references). Most of the existence results for subharmonic solutions assume the (strict) convexity of functional $G$, and few seem to be known in the case that $G$ is not a convex function. In [7], Giannoni considered problem (P) with $G(x)$ replaced by $G(t, x): R \times R^N \to R$ which is not necessarily convex for the second variable and $\lim_{|x| \to \infty} G(t, x) = 0$ uniformly in $t$. Recently, Fonda and Lazer [5] established an existence result without assuming convexity on $G$. They proved the existence of subharmonic solutions of (P) in the case that $G'$ is bounded and $\lim_{|x| \to \infty} G(x) = \infty$. Their argument



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is based on the estimate of critical levels of functionals associated to problem (P). Here we also consider the case that functional $G$ is not necessarily convex, under different assumptions from those of Fonda & Lazer. Instead of making use of the estimate of critical levels of functionals associated to problem (P), we show the existence of subharmonic solutions by applying a Morse index theorem for critical points of min–max type established by Lazer and Solimini [8] (cf. [6] for applications of Lazer and Solimini's results to the convex case). Our approach enables us to treat the case that $G$ is bounded and the case that $G'$ is unbounded.

## 2. Preliminary Notations and the Results

For each $T > 0$, we set

$$H_T = H^{1/2}(S_T^1; R^N),$$

where $S_T^1 = [0, T]/\{0, T\}$. $H_T$ is regarded as a subspace of $L^2([0, T]; R^N)$. $\|\cdot\|_T$ and $\langle\cdot, \cdot\rangle_T$ stand for the norm and inner product of $L^2([0, T]; R^N)$, respectively. For each functional $f \in C^1(H_T, R)$ and $u \in H_T$, we denote by $f'(u)(v)$ the value of the gradient $f'(u): H_T \to R$ at $v \in H_T$. Let $v \in H_T$. We denote by $\bar{v}$ the mean value of $v$, i.e.,

$$\bar{v} = (1/T) \int_0^T v(t)\, dt.$$

We also put $\tilde{v} = v - \bar{v}$. We define subspaces $\tilde{H}_T$ and $\overline{H}_T$ by

$$\tilde{H}_{kT} = \{v \in H_{kT} : \bar{v} = 0\} \qquad \text{and} \qquad \overline{H}_{kT} = \{v \in H_{kT} : \tilde{v} = 0\},$$

respectively. For each $x \in R^N$ and each $n \times n$ matrix $A$, we denote by $|x|$ and $|A|$ the norm of $x$ and the operator norm of $A$, respectively. The inner product of $R^N$ is denoted by $\langle\cdot, \cdot\rangle$. For $G: R^N \to R$, $G''$ denote the Hessian of $G$.

We can now state our main results.

THEOREM 1. *Assume that each critical point of $G$ is nondegenerate and the following conditions are satisfied*:

(G1) $\quad \lim_{|x|\to\infty} G(x)/|x|^2 = 0 \quad$ *and* $\quad \lim_{|y|\to\infty} G(y) > G(x)$ *for all* $x \in R$;

(G2) $\quad \lim_{|x|\to\infty} |G''(x)| = \lim_{|x|\to\infty} |G''(x)|/|G'(x)| = 0.$

*Then there exist $T_0 > 0$ and $\rho > 0$ such that for each $f \in \tilde{H}_T$ with $T < T_0$ and $\|\tilde{f}\|_T < \rho$, problem* (P) *has periodic solutions with minimal period $kT$, for any sufficiently large prime number $k$.*

*Remark* 1. The mapping $G'$ can be unbounded under the hypotheses of Theorem 1. For example, conditions (G1) and (G2) are satisfied if $G(x) = |x|^p$ (or $G(x) = \sum_{i=1}^N |x_i|^p$) $(0 < p < 2)$) for $|x|$ sufficiently large. For each $f \in H_T$, problem (P) can be rewritten as

$$u_{tt} + G'(u) - \bar{f} = \tilde{f}.$$

Then the assertion of Theorem 1 holds for each $f \in H_T$ if the assumptions of Theorem 1 are satisfied with $G$ replaced by $G(x) - \langle \bar{f}, x \rangle$. The nondegeneracy of $G$ and condition (G2) imply implicitly that $G'(x) \neq 0$ if $|x|$ is sufficiently large.

THEOREM 2. *Assume that $G$ satisfies* (G1), (G2). *In addition, assume that*

(G3) *there exists $x_0 \in R^N$ such that*

$$\langle G''(y)x_0, x_0 \rangle > 0 \qquad \text{for all } y \in R^N. \tag{2.1}$$

*Then there exists $T_0 > 0$ such that for each $f \in \tilde{H}_T$ with $T < T_0$, problem* (P) *has periodic solutions with minimal period $kT$, for any sufficiently large prime number $k$.*

*Remark* 2. Condition (G3) is a directional convexity of $G$. That is, for any $z \in R^N$, the functional $G$ is strictly convex on the line $\{z + tx_0 : t \in R\}$.

We can remove the restriction on the period $T$ of $f$ if we impose the boundedness condition on $G'$ instead of $G''$.

THEOREM 3. *Assume that* (G1) *and* (G3) *hold. In addition, assume that*

$$\text{(G2')} \qquad \sup_{x \in R^N} |G'(x)| < \infty \qquad \textit{and} \lim_{|x| \to \infty} |G''(x)|/|G'(x)| = 0.$$

*Then for each $f \in \tilde{H}_T$ with $T > 0$, problem* (P) *has periodic solutions with minimal period $kT$, for any sufficiently large prime number $k$.*

In the following, we assume that $G$ satisfies the assumptions of Theorem 1. For each $x \in R^N$, $\lambda_1(x) \leq \lambda_2(x) \leq \cdots \leq \lambda_N(x)$ denote the eigenvalues of the matrix $G''(x)$. We denote by $x_{(i)}$ the $i$th element of $x \in R^N$. $D(r)$ stands for an open ball in $R^N$ centered at 0 with radius $r$. From (G1), we can see that there exists a critical point of $G$. Since each critical point of $G$ is nondegenerate and the set of critical points is bounded by assumption (G2), we can see that the set of critical points of $G$ consists of finite elements.

Let $\{z_1, ..., z_m\} \subset R^N$ be the set of critical points of $G$. Then there exists $c_0 > 0$ such that

$$|\lambda_j(z_i)| > 2c_0 \qquad \text{for } 1 \leq i \leq m \text{ and } 1 \leq j \leq N.$$

For each $1 \leq i \leq m$ and $1 \leq j \leq N$, we denote by $w_{i,j}$ a normalized eigenvector corresponding to the eigenvalue $\lambda_j(z_i)$ of $G''(z_i)$. Then it follows that

$$|\langle G''(z_i)w_{i,j}, w_{i,j}\rangle \geq 2c_0 \qquad \text{for } 1 \leq i \leq m \text{ and } 1 \leq j \leq N. \tag{2.2}$$

Let $x, y \in R^N$. Then we have

$$G'(y) - G'(x) = \int_0^{|y-x|} G''\left(x + t\left(\frac{y-x}{|y-x|}\right)\frac{y-x}{|y-x|}\right) dt. \tag{2.3}$$

On the other hand, we have by condition (G2) that for any $\delta > 0$, there exists $d_0(\delta) > 0$ such that

$$|G''(z)| \leq \delta|G'(z)| \qquad \text{for all } z \in R^N \text{ with } |z| \geq d_0(\delta). \tag{2.4}$$

Now let $\varepsilon > 0$ and let $d(\varepsilon) > 0$ be so large that $d(\varepsilon) > d_0(\delta) + \varepsilon$, where $\delta > 0$ satisfies $(e^{\delta\varepsilon} - 1) = 1/2N$. Let $x, y \in R^N$ with $|x| > d(\varepsilon)$ and $y \in x + D(\varepsilon)$. Then since the line segment connecting $x$ and $y$ is contained in the set $\{z \in R^N : |z| > d_0(\delta)\}$, we find by (2.3) and (2.4) that

$$|G'(y)| \leq |G'(x)| + \delta\int_0^{|y-x|}\left|G'\left(x + t\left(\frac{y-x}{|y-x|}\right)\right)\right| dt.$$

It then follows from Gronwall's inequality that

$$|G'(y)| \leq e^{\delta|y-x|}|G'(x)|. \tag{2.5}$$

Then by (2.3), (2.4), and (2.5),

$$|G'(y) - G'(x)| \leq \delta\int_0^{|y-x|} e^{\delta t}|G'(x)|\,dt \leq (e^{\delta\varepsilon} - 1)|G'(x)| = |G'(x)|/2N$$

for all $x \in R^N$ with $|x| > d(\varepsilon)$ and $y \in x + D(\varepsilon)$. On the other hand, we have that for each $x \in R^N$, there exists $i$ such that $|G'(x)_{(i)}| \geq |G'(x)|/N$.

Then it follows from the observation above that for each $\varepsilon > 0$, there exists $i \in \{1, \ldots, N\}$ such that

$$G'(x)_{(i)}G'(y)_{(i)} > 0 \qquad \text{for all } x \in R^N \text{ with } |x| > d(\varepsilon) \text{ and } y \in x + D(\varepsilon). \tag{2.6}$$

On the other hand, it follows from (2.2) that there exists $\varepsilon_0 > 0$ such that for each $1 \leq i \leq m$,

$$|G''(x) - G''(z_i)| < c_0/2(N-1) \qquad \text{and} \qquad |\langle G''(x)w_{i,j}, w_{i,j}\rangle \geq c_0 \tag{2.7}$$

for all $1 \leq j \leq N$ and $x \in z_i + D(\varepsilon_0)$. We set

$$m_0 = \inf\{|G'(x)| : x \in R^N, |x| \leq d(\varepsilon_0), x \notin \bigcup (z_i + D(\varepsilon_0/2))\}. \tag{2.8}$$

It is obvious from the assumption that $m_0 > 0$.

Let $E$ be a Banach space and $E = Y \oplus Z$, where $Y$ and $Z$ are subspaces of $E$ with $\dim Y = k$. Let $F: E \to R$ be a $C^2$ functional. Let $u \in E$ be a critical point of $F$. If there exists a maximal integer $q$ such that $F''(u): E \to E$ is negative definite (semi-negative definite) on some $q$-dimensional subspace of $E$, then the Morse index (augmented Morse index) of $u$ is $q$. In the following, the Palais–Smale condition (cf. [10]) is referred to as (P-S). Now suppose that $F$ satisfies the following conditions:

(S) $\inf_{x \in Z} F(z) > d > -\infty$ and $\lim_{\|y\| \to \infty} F(y) < d$, for $y \in Y$.

It then follows from (S) that

(I) there exists $r > 0$ such that $\max_{y \in S(r)} F(y) < \inf_{z \in Z} F(z)$,

where $S(r) = \{y \in Y : \|y\| = r\}$. Let $\bar{\Gamma}$ be the family of sets $A \subset E$ satisfying the following properties:

(C1) For any continuous mapping $\sigma: A \to Y$ such that $\sigma(x) = x$ for all $x \in A \cap S$, the condition $0 \in \sigma(A)$ holds;

(C2) there exists a constant $d$ such that for each integer $n \geq 1$, $A$ can be covered by $n^k$ balls of radius $d/n$.

We set

$$\bar{c} = \inf_{A \in \bar{\Gamma}} \max_{y \in A} F(y).$$

In [8], Lazer and Solimini established the following theorem, which we use to prove our main results.

THEOREM A. *Assume* (I). *If the* (P-S) *condition holds in* $\bar{c}$ *and if* $F'$ *is Fredholm of index* 0 *at all the critical points in* $F^{-1}(\bar{c})$, *then at least one of them has Morse index less than or equal to* $k$ *and augmented Morse index greater than or equal to* $k$.

*Remark* 3. Let $z_0 \in Z$. Then the assertion of Theorem A holds if (S) and (I) are replaced by

$$\text{(S}'\text{)}\quad \inf_{x \in Z} F(z) > d > -\infty \text{ and } \lim_{\|y\| \to \infty} F(y) < d, \text{ for } y \in z_0 + Y$$

and

$$\text{(I}'\text{)}\quad \text{there exists } r > 0 \text{ such that } \max_{y \in S(r) + z_0} F(y) < \inf_{z \in Z} F(z).$$

In fact, if (S′) and (I′) are satisfied by $F$, the functional $F_0$ defined by $F_0(u) = F(u + z_0)$ satisfies (S) and (I).

## 3. PROOFS OF THE MAIN RESULTS

We put $\max G'' = \max\{|G''(x)| : x \in R^N\}$. Let $T_0 = 2/\sqrt{\max G''}$. In the following, we fix a positive number $T$ such that $T < T_0$. Let $f \in \tilde{H}_T$. We put

$$S = \{u \in H_T : u \text{ is a solution of problem (P)}\}.$$

LEMMA 1. (1) *$S$ is bounded in $H_T$;*

(2) *there exists $\rho > 0$ such that if $\|f\|_T \leq \rho$, then for each $u \in S$, there exists $i \in \{1, 2, ..., m\}$ satisfying*

$$|G''((u(t)) - G''(z_i))| < \frac{c_0}{2(N-1)} \qquad \textit{and} \qquad |\langle G''(u(t))w_{i,j}, w_{i,j}\rangle| \geq c_0$$

*for all $t \in [0, T]$ and $1 \leq j \leq N$.*

*Proof.* Let $u \in H_T$ be a solution of (P). We multiply (P) by $\tilde{u}$ and integrate over $[0, T]$. Then noting that $f \in \tilde{H}_T$ and $G'(u) = G'(\bar{u}) + G''(\theta)(\tilde{u})$ for some mapping $\theta : [0, T] \to R^N$, we have

$$\|u_t\|_T^2 = \langle G'(u) - f, \tilde{u}\rangle_T \leq (\max G'')\|\tilde{u}\|_T^2 + \|f\|_T\|\tilde{u}\|_T. \tag{3.1}$$

Then since $\|\bar{u}\|_T \le (T/2\pi)\|u_t\|_T \le (T/2)\|u_t\|_T$, we have that

$$\left(1 - \frac{T^2}{4}\max G''\right)\|u_t\|_T^2 \le \|f\|_T\|\bar{u}\|_T \le \frac{T}{2}\|u_t\|_T\|f\|_T,$$

and then

$$\|\bar{u}\|_T \le \frac{T}{2}\|u_t\|_T \le C\|f\|_T, \tag{3.2}$$

where $C = 1/((4/T^2) - \max G'')$. We next multiply (P) by $G'(u)$. Then we have by (3.1) and (3.2) that

$$\begin{aligned} \|G'(u)\|_T^2 &\le \langle f, G'(u)\rangle_T + \|G''(u)u_t\|_T\|u_t\|_T \\ &\le ((\max G'')\|\bar{u}\|_T\|f\|_T + (\max G'')\|u_t\|_T^2 \le C_1\|f\|^2, \end{aligned} \tag{3.3}$$

where $C_1$ is a constant depending only on $C$ and $\max G''$. It follows from (3.1) and (3.2) that there exists $\varepsilon > 0$ such that

$$|u(t) - \bar{u}| \le \varepsilon \qquad \text{for all } t \in [0, T]. \tag{3.4}$$

Then we have that $|\bar{u}| < d(\varepsilon)$. In fact, if $|\bar{u}| \ge d(\varepsilon)$, then we have by (2.6) that there exists $1 \le i \le N$ such that

$$G'(u(t))_{(i)} \cdot G''(\bar{u})_{(i)} > 0 \qquad \text{for all } t \in [0, T].$$

This implies that

$$0 \ne \int_0^T G'(u) = \int_0^T f\,dt = 0.$$

This is a contradiction. Thus we have seen that assertion (1) holds. We next prove (2). By (3.2) and (3.3), we can choose $\rho > 0$ so small that if $\|f\|_T \le \rho$, each solution $u \in H_T$ of (P) satisfies

$$|u(t) - \bar{u}| \le \varepsilon_0/2 \qquad \text{for all } t \in [0, T] \tag{3.5}$$

and

$$|G'(u(t))| \le m_0/2 \qquad \text{for all } t \in [0, T]. \tag{3.6}$$

Now suppose that $\|f\|_T \le \rho$. From the argument above, we have that $|\bar{u}| \le d(\varepsilon_0)$. If $\bar{u} \notin \cup (z_i + D(\varepsilon_0/2))$, then by (2.8) and (3.5), we find that $\sup_{t\in[0,T]} |G'(u(t))| \ge m_0$. This contradicts (3.6). Thus we have seen that $\bar{u} \in \cup (z_i + D(\varepsilon_0/2))$. Then assertion (2) follows from (3.5) and (2.7). ∎

Throughout the rest of this section, we assume that $\|f\|_T \le \rho$. For each positive integer $k$, we define a functional $F_k : H_{kT} \to R$ by

$$F_k(u) = \int_0^{kT} \left(\frac{1}{2}|u_t|^2 - G(u) + \langle f(t), u(t)\rangle\right) dt \qquad \text{for } u \in H_{kT}.$$

It then easy to see that each critical point of $F_{kT}$ is a $kT$-periodic solution of (P). We denote by m-index$_k(u)$ (am-index$_k(u)$) the Morse index (augmented Morse index) of $F_k$ at a critical point $u \in H_{kT}$ of $F_k$. Let $\Gamma_k$ be the set $\overline{\Gamma}$ defined in Section 2 with $E$, $Y$, and $Z$ replaced by $H_{kT}$, $\overline{H}_{kT}$, and $\tilde{H}_{kT}$, respectively. We set

$$c_k = \inf_{A\in\Gamma_k} \max_{y\in A} F_k(y).$$

LEMMA 2. *For each $k \ge 1$, $F_k$ satisfies* (I′) *with $E$, $Y$, and $Z$ replaced by $H_{kT}$, $\overline{H}_{kT}$, and $\tilde{H}_{kT}$.*

*Proof.* Let $k \ge 1$. Let $u_0 \in \tilde{H}_T$ be the unique element satisfying

$$m_1 = \langle f, u_0\rangle_T + \frac{1}{2}\|u_{0t}\|_T^2 = \min\left\{\langle f, v\rangle_T + \frac{1}{2}\|v_t\|_T^2 : v \in \tilde{H}_T\right\}.$$

Then since $f \in \tilde{H}_T$, we find that

$$m_1 k = \min\left\{\langle f, v\rangle_{kT} + \frac{1}{2}\|v_t\|_{kT}^2 : v \in \tilde{H}_{kT}\right\}.$$

For each $r > 0$, we set

$$S_0(r) = \{x + u_0 : x \in R^N, |x| = r\}.$$

Then it follows from (G1) and the definition of $u_0$ that

$$\limsup_{r\to\infty}\{F_k(v) : v \in u_0 + R^N\backslash D(r)\} = M_k = -\left(\lim_{|x|\to\infty} G(x)\right) kT + m_1 k. \quad (3.7)$$

Here let $C$ be a positive number such that

$$\|\tilde{v}\|_{kT} \le C\|v_t\|_{kT} \qquad \text{for } v \in H_{kT}.$$

By (G1), we can choose $c_1, c_2 > 0$ such that $c_1C^2 < 1/2$ and

$$|G(x)| \le c_1|x|^2 + c_2 \qquad \text{for all } x \in R^N.$$

Then we have by (G1) and the inequalities above that

$$\begin{aligned} F_k(v) &= \int_0^{kT} \left(\frac{1}{2}\|v_t\|^2 - G(v) + \langle f, v\rangle\right) dt \\ &> \left(\left(\frac{1}{2} - c_1C^2\right)\|v_t\|_{kT}^2 - C\|f\|_{kT}\|v_t\|_{kT} - c_2kT\right) \end{aligned}$$

for $v \in \tilde{H}_{kT}$. Then there exists $d > 0$ such that

$$\inf\{F_k(v) : v \in \tilde{H}_{kT}, \|v_t\| \ge d\} > M_k. \tag{3.8}$$

On the other hand, noting that $\sup\{|v(t)| : v \in \tilde{H}_{kT}, \|v_t\| \le d\} < \infty$, we can see that

$$\sup\left\{\int_0^{kT} G(v) : v \in \tilde{H}_{kT}, \|v_t\| \le d\right\} = M_d < \left(\lim_{|x|\to\infty} G(x)\right) kT.$$

Then we have

$$F_k(v) = \int_0^{kT} \left(\frac{1}{2}\|v_t\|^2 - G(v) + \langle f, v\rangle\right) dt > -M_d - m_1k \tag{3.9}$$

for all $v \in \tilde{H}_{kT}$ with $\|v_t\| \le d$. Consequently, we have by (3.8) and (3.9) that

$$\inf\{F_k(v) : v \in \tilde{H}_{kT}\} = \tilde{M}_k > M_k. \tag{3.10}$$

Therefore $F_k$ satisfies (S′) with $E$, $Y$, and $Z$ replaced by $H_{kT}$, $\overline{H}_{kT}$, and $\tilde{H}_{kT}$ and $z_0 = u_0$. Then for each $k \ge 1$, there exists $r_k > 0$ such that $\sup\{F_k(v) : v \in S_0(r_k)\} < \tilde{M}_k$. That is $F_k$ satisfies (I′) with $S(r)$ replaced by $S_0(r_k)$. ∎

LEMMA 3. *For each $k \geq 1$, the following assertions hold.*

(i) $c_k > M_k$;

(ii) *$F_k$ satisfies* (P-S) *at $c_k$.*

*Proof.* Let $k \geq 1$. (i) Let $P$ be the projection from $H_{kT}$ onto $\overline{H}_{kT}$. Then from the definition of $\Gamma_k$, $0 \in P(A)$ for any $A \in \Gamma_k$. This implies that $A \cap \tilde{H}_{kT} \neq \varnothing$ for any $A \in \Gamma_k$. Then (i) follows from (3.10). (ii) Let $\{u_n\} \in H_{kT}$ be a sequence such that $F_k'(u_n) \to 0$ and $F_k(u_n) \to c_k$, as $n \to \infty$. Let $C$ be the positive constant defined in the proof of Lemma 2. Then we have

$$\begin{aligned} F_k(u_n) &= \int_0^{kT} \left( \frac{1}{2} \|\tilde{u}_{nt}|^2 - G(u_n) + \langle f(t), \tilde{u}_n(t) \rangle \right) dt \\ &\geq (1/2C)\|\tilde{u}_n\|_{kT}^2 - \int_0^{kT} G(u_n)\, dt - \|f\|_{kT}\|\tilde{u}_n\|_{kT}. \end{aligned} \tag{3.11}$$

Suppose that $\limsup_n \|\tilde{u}_n\|_{kT} = \infty$. If there exists $C_1 > 0$ and

$$\limsup_n \|\overline{u}_n\|_{kT}/\|\tilde{u}\|_{kT} = C_1 < \infty,$$

then nothing that $\|u_n\|_{kT} \leq (2C_1 + 1)\|\tilde{u}_n\|$ for $n$ sufficiently large, and $\lim_{|x|\to\infty} G(x)/|x|^2 = 0$, we have from (2.11) that $\limsup F_k(u_n) = \infty$. This is a contradiction. Therefore we have that $\limsup_n \|\overline{u}_n\|_{kT}/\|\tilde{u}_n\|_{kT} = \infty$. From the assumption, we have by applying $F_k'(u_n)$ to $\tilde{u}_n/\|\tilde{u}_n\|_{kT}$ that

$$\lim_{n\to\infty} \left( \|\tilde{u}_{nt}\|_{kT}^2/\|\tilde{u}_n\|_{kT} - \int_0^{kT} \langle G'(u_n) - f, \tilde{u}_n/\|\tilde{u}_n\|_{kT} \rangle\, dt \right) = 0. \tag{3.12}$$

On the other hand, for each $n \geq 1$, there exists a mapping $\theta_n : [0, kT] \to R^N$ such that

$$\theta_n(t) \in \{\overline{u}_m(t) + s\tilde{u}_n(t) : s \in [0, 1]\} \qquad \text{for each } t \in [0, kT]$$

and

$$G'(u_n) = G'(\overline{u}_n) + G''(\theta_n)\tilde{u}_m \qquad \text{for each } n \geq 1.$$

Then since $\tilde{u}_n$ has mean value zero, we have that

$$\int_0^{kT} \langle G'(u_n), \tilde{u}_n \rangle\, dt = \int_0^{kT} \langle G''(\theta_n)\tilde{u}_n, \tilde{u}_n \rangle\, dt. \tag{3.13}$$

Since $\lim_{n\to\infty} \|\bar{u}_n\|_{kT}/\|\tilde{u}_n\|_{kT} = \infty$, we have that $\lim_{n\to\infty} |\theta_n(t)|_{kT} = \infty$ on $[0, kT]$ and then $\lim_{n\to\infty} G''(\theta_n(t)) = 0$ for all $t \in [0, kT]$. Therefore recalling that $\|\tilde{u}_n\|_{kT} \le C\|\tilde{u}_{nt}\|$. we have by (3.13) that

$$\begin{aligned}
&\lim_{n\to\infty} \left( \|\tilde{u}_{nt}\|_{kT}^2/\|\tilde{u}_n\| - \int_0^{kT} \langle G'(u_n) - f, \tilde{u}_n/\|\tilde{u}_n\|_{kT}\rangle \, dt \right) \\
&\quad \ge \lim_{n\to\infty} \left( (1/C^2)\|\tilde{u}_n\|_{kT} - \int_0^{kT} \langle G'(\theta_n) - f, \tilde{u}_n/\|\tilde{u}_n\|_{kT}\rangle \, dt \right) \\
&\quad \ge \lim_{n\to\infty} \left( (1/C^2)\|\tilde{u}_n\|_{kT} - \int_0^{kT} \langle G'(\theta_n), \tilde{u}_n/\|\tilde{u}_n\|_{kT}\rangle \, dt - \|f\|_{kT} \right).
\end{aligned}$$

This contradicts (3.12). Consequently, we obtain that $\{\tilde{u}_n\}$ is bounded in $H_{kT}$. We next see that $\{\bar{u}_n\}$ is bounded. Suppose that $\{\bar{u}_n\}$ is unbounded. We may assume by extracting a subsequence that $\lim \|\bar{u}_n\| = \infty$. It then follows that

$$\lim_{n\to\infty} \|G'(u_n)\|_{kT} = 0. \tag{3.14}$$

In fact, if $\limsup \|G'(u_n)\| > 0$, then recalling that $\{\tilde{u}_n\}$ is bounded, we have by (2.6) that

$$\limsup_n \left| \int_0^{kT} G'(u_n) \, dt \right| > 0.$$

Since

$$\lim_{n\to\infty} F_k'(u_n)(-1) = \lim_{n\to\infty} \int_0^{kT} (G'(u_n) - f) \, dt = \lim_{n\to\infty} \int_0^{kT} G'(u_n) \, dt = 0,$$

this is a contradiction. Here we define a convex functional $F_0$ by

$$F_0(v) = \int_0^{kT} \left( \frac{1}{2}\|v_t\|^2 + \langle f, v\rangle \right) dt \qquad \text{for } v \in H_{kT}.$$

Then we have that

$$\begin{aligned}
F_k'(u_n)(v) &= \int_0^{kT} (\langle u_{nt}, v_t\rangle - \langle G'(u_n), v\rangle + \langle f, v\rangle) \, dt \\
&= F_0'(u_n)(v) - \int_0^{kT} \langle G'(u_n-), v\rangle \, dt \qquad \text{for } v \in H_{kT}.
\end{aligned}$$

Recalling that $F'_k(u_n) \to 0$ as is $n \to \infty$, we find by (3.14) and the equality above that $F'_0(u_n) \to 0$ as $n \to \infty$. It is then easy to see from the definition of functional $F_0$ that $\{u_n\}$ is a minimizing sequence of the functional $F_0$. This implies that

$$\lim_{n\to\infty}\left(\langle f, u_n\rangle_{kT} + \frac{1}{2}\|u_{nt}\|_{kT}^2\right) = k\cdot m_1.$$

Therefore from (G1), we obtain that

$$\lim_{n\to\infty} F_k(u_n) = \lim_{n\to\infty}\left(\frac{1}{2}\|u_{nt}\|_{kT}^2 + \langle f, u_n\rangle_{kT} - \left(\lim_{|x|\to\infty} G(x)\right)kT\right) = M_k < c_k.$$

This contradicts the assumption. Thus we obtain that $\{\bar{u}_n\}$ is bounded. Then we can see by the standard argument (cf. [10]) that $\{u_n\}$ has a convergent subsequence. This completes the proof. ∎

*Remark* 4. The nondegeneracy of $G$ is used only for proving (2) of Lemma 1. Then (1) of Lemma 1, Lemma 2, and Lemma 3 are valid without the nondegeneracy.

Let $u \in S$. Then by (2) of Lemma 1, there exists $1 \le i \le m$ such that

$$|\langle G''(u(t))w_{i,j}, w_{i,j}\rangle| \ge c_0 \qquad \text{for all } t \in [0, T] \text{ and } 1 \le j \le N.$$

Then since $G \in C^2(R^N, R)$ and $u : [0, T] \to R^N$ is continuous, this implies that for each $u \in S$, there exists $1 \le i \le m$ such that for each $1 \le j \le N$,

$$\langle G''(u(t))w_{i,j}, w_{i,j}\rangle \ge c_0 \qquad \text{on } [0, T]$$

or

$$\langle G''(u(t))w_{i,j}, w_{i,j}\rangle \le -c_0 \qquad \text{on } [0, T]$$

holds. Here we put

$$\bar{\mu}_i(u) = \max_{1\le j\le N} \sup\{\langle G''(u(t))w_{i,j}, w_{i,j}\rangle, t \in [0, T]\}$$

and

$$\underline{\mu}_i(u) = \max_{1\le j\le N} \inf\{\langle G''(u(t))w_{i,j}, w_{i,j}\rangle, t \in [0, T]\}$$

for each $1 \le i \le m$. We also set

$$S_1 = \{u \in S : \underline{\mu}_i(u) \ge c_0 \text{ for some } 1 \le i \le m\}$$

$$S_2 = \{u \in S : \overline{\mu}_i(u) \le -c_0 \text{ for some } 1 \le i \le m\}.$$

Then it follows from the observation above that $S = S_1 \cup S_2$.

PROPOSITION 1. *There exists a positive integer $k_0$ such that for each $k \ge k_0$,*

$$m\text{-}index_k(u) > N \qquad \textit{for all } u \in S_1.$$

*Proof.* Let $u \in S_1$. Then from the definition of $S_1$, we have that there exist $1 \le i \le m$ and $1 \le j \le N$ such that

$$\langle G''(u(t)w_{i,j}, w_{i,j}\rangle \ge c_0 \qquad \text{for all } t \in [0, T].$$

Here we put $w = w_{i,j}$. Then from the inequality above

$$(1/T)\int_0^T \langle G''(u(t))w, w\rangle \ge c_0.$$

Here we put $d_{mn} = (1/T)\int_0^T (G''(u(t)))_{mn}\, dt$ for each $1 \le m, n \le N$, where $(G''(u(t)))_{mn}$ is the $(m, n)$-element of the matrix $G''(u(t))$. Then from the inequality above, we find that the matrix $D = \{d_{mn}\}$ satisfies

$$\langle Dw, w\rangle \ge c_0. \tag{3.15}$$

We note that $D$ is Hermitian. We define a matrix $A = \{a_{mn}\}$ by $a_{mn}(t) = (G''(u(t)))_{mn} - d_{mn}$ for $m, n \in \{1, ..., N\}$. Then each $a_{mn}$ is a $T$-periodic function with $\bar{a}_{mn} = 0$. From the definitions of $D$ and $A$, we have

$$G''(u) = D + A.$$

By (3.15), $D$ has a positive eigenvalue $\lambda$ ($\ge c_0$). Let $y = (y_1, ..., y_N) \in R^N$ be a normalized eigenvector corresponding to the eigenvalue $\lambda$. Here we choose a positive prime number $k_0$ such that

$$(2\pi(N+1)/k_0T)^2 < \lambda \qquad \text{and} \qquad 2(N+1) < k_0. \tag{3.16}$$

Now let $k \geq k_0$, and

$$v_i(t) = \left(\sqrt{\frac{2}{kT}}\sin\left(\frac{2\pi i}{kT}t\right)\right)y \qquad \text{for } 1 \leq i \leq N+1.$$

Then since each $a_{mn}$ is $T$-periodic and $\bar{a}_{mn} = 0$, we have that each $n$th element of $Av_i$ consists of the linear combination of functions $\sin(2\pi(kj \pm i)t/kT)$, $\cos(2\pi(kj \pm i)t/kT)$, $j = 1, 2, \ldots$. From the second inequality in (3.16), we find that these functions are orthogonal to $\sin((2\pi qt/kT))$, $1 \leq q \leq N + 1$ in $L^2(0, kT)$. It then follows that

$$\langle Av_i, v_j\rangle_{kT} = 0 \qquad \text{for all } 1 \leq i, j \leq N+1.$$

Then we find that

$$\begin{aligned}\langle -v_{itt} - G''(u)v_i, v_i\rangle_{kT} &= (2\pi i/kT)^2 - \langle Dv_i, v_i\rangle_{kT} \\ &= (2\pi i/kT)^2 - \langle Dy, y\rangle \\ &= (2\pi i/kT)^2 - \lambda < 0\end{aligned} \tag{3.17}$$

for all $1 \leq i \leq N+1$. On the other hand, recalling that $2(N+1) < k$ and each $a_{mn}$ is $T$-periodic, we can see that

$$\langle G''(u)v_i, v_j\rangle_{kT} = \langle (A + D)v_i, v_j\rangle_{kT} = 0 \qquad \text{for } i \neq j.$$

Then it easy to see from (3.17) and the equality above that for any $v \in \operatorname{span}\{v_1, \ldots, v_{N+1}\}\backslash\{0\}$,

$$\langle -v_{tt} - G''(u)v, v\rangle_{kT} < 0.$$

This implies that m-index$_k(u) > N$ for $k \geq k_0$. ∎

PROPOSITION 2. *For each* $k \geq 1$,

$$\textit{am-index}_k(u) = 0 \qquad \textit{for all } u \in S_2.$$

*Proof.* Let $u \in S_2$. That is, there exists an orthonormal basis $\{w_{i,1}, w_{i,2}, \ldots, w_{i,N}\}$ of $R^N$ such that

$$\langle G''(u(t))w_{i,j}, w_{i,j}\rangle \leq c_0 \qquad \text{on } [0, T] \text{ for all } 1 \leq j \leq N. \tag{3.18}$$

Since $|G''(u(t)) - G''(z_i)| < c_0/2(N-1)$ on $[0, T]$ by the first inequality of (2) of Lemma 1, we find that for each $j, k \in \{1, \ldots, N\}$ with $i \neq j$,

$$\begin{aligned}
&|\langle G''(u(t))w_{i,j}, w_{i,k}\rangle| \\
&\quad \le |\langle (G''(u(t)) - G''(z_i))w_{i,j}, w_{i,k}\rangle + \langle G''(z_i)w_{i,j}, w_{i,k}\rangle| \qquad (3.19)\\
&\quad \le |G''(u(t)) - G''(z_i)||w_{i,j}||w_{i,k}| \le c_0/2(N-1).
\end{aligned}$$

Then it follows from (3.18) and (3.19) that

$$\langle G''(u(t))w, w\rangle \le -c_0/2 \qquad \text{for all } w \in R^N \text{ with } |w| = 1 \text{ and } t \in [0, T].$$

Then we have that

$$\langle -v_{tt} - G''(u)v, v\rangle_{kT} \ge \|v_t\|_{kT}^2 + (c_0/2)\|v\|_{kT}^2 \qquad \text{for all } v \in H_{kT}.$$

Then the assertion follows. ∎

Now we can complete the proof of Theorem 1.

*Proof of Theorem* 1. Let $k$ be a prime number with $k \ge k_0$. Then by applying Theorem A to the functional $F_k$, we find that there exists a critical point $u_k$ of $F_k$ satisfying that

$$\text{am-index}_k(u) \ge N \qquad \text{and} \qquad \text{m-index}_k(u) \le N.$$

Since $k$ is a prime number, the minimal period of $u_k$ is $T$ or $kT$. If the minimal period of $u_k$ is $T$, we have that $u_k \in S = S_1 \cup S_2$. Then by Proposition 1 and Proposition 2, we find that m-index$_k(u) > N$ or am-index$_k(u) = 0$ holds. This is a contradiction. Therefore the minimal period of $u_k$ is $kT$. This completes the proof. ∎

*Proof of Theorem* 2. By Remark 4, we have that $S$ is bounded in $H_T$ by (1) of Lemma 1. Then we have by (G3) that there exists $d > 0$ such that

$$\langle G''(u(t))x_0, x_0\rangle \ge d \qquad \text{for all } u \in S \text{ and } t \in [0, T].$$

Then the argument of Proposition 1 is valid for all $u \in S$ with $w \in R^N$ and $c_0$ replaced by $x_0$ and $d$, respectively. That is, we have that there exists a positive integer such that for each $k \ge k_0$, m-index$_k(u) > N$ for all $u \in S$. Then the assertion of Theorem 2 follows from the argument of the proof of Theorem 1. ∎

*Proof of Theorem* 3. The restriction that $T < T_0$ is used only in inequality (3.2) to obtain the boundedness of $\tilde{S} = \{\tilde{u} : u \in S\}$. If $G'$ is bounded, then $\tilde{S}$ is bounded for any $T > 0$ (see the proof of Theorem 1 of [5]). Then the assertion follows by the argument for Theorem 1 and Theorem 2. ∎


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